Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Linear Algebra and its Applications 407 (2005) 1–10

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Nonstationary multisplittings with general weighting matrices for mildly nonlinear systems ☆

Chuan-Long Wang [a,*], Guo-Yan Meng [b]

[a] *Department of Mathematics, Taiyuan Teachers' College, Taiyuan 030012, PR China*
[b] *Department of Computer Science, Xinzhou Teachers' College, PR China*

Received 10 December 2003; accepted 25 April 2005
Available online 26 July 2005
Submitted by M. Neumann

Abstract

In this paper, we present a new nonstationary multisplitting iterative method for solving the mildly nonlinear equations $Ax = F(x)$, where $A \in R^{n\times n}$ is an n-by-n sparse real symmetric positive definite matrix and $F : R^n \to R^n$ is a general nonlinear mapping, and establish the convergence theories with general weighting matrices by more than one inner iteration. Meanwhile, we also give the lower bound of the inner iteration number in this nonstationary multisplitting iterative method.

AMS classification: 65F10; 65F50

Keywords: System of mildly nonlinear equations; Symmetric positive definiteness; Nonstationary multisplitting; General weighting matrices

☆ This work is supported by academic youth ring leader of Shanxi province.
* Corresponding author.
E-mail address: clwang@public.ty.sx.cn (C.-L. Wang).

doi:10.1016/j.laa.2005.04.029

1. Introduction and preliminaries

For the mildly nonlinear equations $Ax = F(x)$, where $A \in R^{n\times n}$ is an n-by-n sparse real matrix and $F : R^n \to R^n$ is a general nonlinear mapping, Bai et al. [3–5] presented a class of parallel multisplitting two-stage (MTS) iterative methods, and obtained its local and global convergence properties by using weak regular splittings when the system matrix A is a monotone matrix or an H-matrix and the nonlinear mapping F is P-bounded in the sense of wise ordering. Based on the results in [11], Bai and Wang [12] studied the local and global convergence properties of MTS iterative method by using symmetric P-regular splittings when the system matrix A is a symmetric positive definite matrix and the nonlinear mapping F is S-bounded in the sense of positive definite ordering. But the weighting matrices of the MTS iterative method in [12] are scalar matrices. As shown in [6,9,10], the MTS iterative method has little applicability for analysis of parallel processing when the weighting matrices are scalar matrices. In order to eliminate this restrictive condition, some special techniques and methods were discussed (see [7–10,13,15]). In this paper, we discuss a new nonstationary multisplitting iterative method for the mildly nonlinear equations, it is assumed that more than one inner iteration is performed in each processor as shown in [9,10,13].

This paper is arranged as follows: We give some notations and preliminaries in this section, the nonstationary multisplitting iterative method for this nonlinear equations is constructed in Section 2. Finally, the convergence properties of the nonstationary multisplitting iterative method with general weighting matrices are discussed in Section 3.

Some notations and concepts employed in the subsequent discussion are as follows: x^{T} and A^{T} represent the transpose of a vector x and a matrix A, respectively. $\|A\|_2$ denotes the Euclidean norm of the matrix $A \in R^{n\times n}$, and $A \succ 0 (\succeq 0)$ means that $A \in R^{n\times n}$ is a symmetric positive definite (semi-definite) matrix. If $A \in R^{n\times n}$ is a symmetric matrix, then there exists an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q^{\mathrm{T}}\Lambda Q$. Therefore, we can define the comparison matrix $\langle A\rangle$ of the matrix A by $\langle A\rangle = Q^{\mathrm{T}}|\Lambda|Q$. Obviously, $\langle A\rangle \succeq 0$. For A, B, we say $A \succ B (A \succeq B)$ if $A - B \succ 0 (A - B \succeq 0)$ holds. $A = M - N$ is called a splitting of the matrix $A \in R^{n\times n}$ if $M \in R^{n\times n}$ is nonsingular. This splitting is called a convergent splitting if $\rho(M^{-1}N) < 1$, a P-regular splitting if $M + N$ is positive definite, and a symmetric P-regular splitting if $M \succ 0$ and $N \succeq 0$. Evidently, a symmetric P-regular splitting is a P-regular splitting (see [1,12,14]). Finally, $[m]$ denotes the integer part of real number m.

2. The nonlinear nonstationary multisplitting iterative method

For $A \in R^{n\times n}$, let $A = M_i - N_i$, $i = 1, 2, \ldots, \alpha$, be its α splittings and E_i, $i = 1, 2, \ldots, \alpha$, be α nonnegative diagonal matrices satisfying $\sum_{i=1}^{\alpha} E_i = I$ (the n-by-n

identity matrix). Then we call the collection of triples $(M_i, N_i, E_i)_{i=1}^{\alpha}$ a multisplitting of the matrix A [6]. For the large sparse systems of mildly nonlinear equations

$$Ax = F(x), \tag{2.1}$$

we construct a new nonstationary multisplitting iterative method, which is different from those in [3–5].

Method 2.1 (*The nonlinear nonstationary multisplitting iterative method*). Given any initial vector $x^{(0)} \in R^n$, $k = 1, 2, \ldots,$ till convergence.

For $i = 1$ to α

$$y_i^{(0)} = x^{(k-1)}$$

For $j = 0$ to $s(i, k) - 1$

$$M_i y_i^{(j+1)} = N_i y_i^{(j)} + F(y_i^{(j)}), \quad i = 1, 2, \ldots, \alpha, \tag{2.2}$$

$$x^{(k)} = \sum_{i=1}^{m} E_i y_i^{(s(i,k))}. \tag{2.3}$$

For the nonlinear mapping $F : R^n \to R^n$, there exists a symmetric matrix $S(x, y)$: $R^n \times R^n \to R^{n\times n}$ such that

$$F(x) - F(y) = S(x, y)(x - y), \quad \forall x, y \in R^n. \tag{2.4}$$

An example of such matrix $S(x, y)$ is obtained by

$$S(x, y) = \begin{cases} \frac{(F(x)-F(y))(F(x)-F(y))^{\mathrm{T}}}{(F(x)-F(y))^{\mathrm{T}}(x-y)}, \\ \quad \text{if } |(F(x) - F(y))^{\mathrm{T}}(x - y)| \geqslant \frac{1}{2}(x - y)^{\mathrm{T}}(x - y), \\ I + \frac{(F(x)-F(y)-x+y)(F(x)-F(y)-x+y)^{\mathrm{T}}}{(F(x)-F(y)-x+y)^{\mathrm{T}}(x-y)}, \\ \quad \text{if } |(F(x) - F(y))^{\mathrm{T}}(x - y)| < \frac{1}{2}(x - y)^{\mathrm{T}}(x - y), \end{cases} \tag{2.5}$$

which follows from a quasi-Newton update [2] with respect to the nonlinear mapping F.

3. Convergence analysis

In order to establish the convergence theories for Method 2.1, we make the following three basic assumptions about the system of mildly nonlinear equations (2.1):

Assumption (i) The matrix A is symmetric positive definite;
Assumption (ii) For the matrix $S(x, y)$ satisfying (2.4), there exists a symmetric positive definite matrix S such that $\langle S(x, y)\rangle \preceq S$ holds for all $x, y \in R^n$;
Assumption (iii) $A - S \succ 0$.

The nonlinear mapping $F : R^n \to R^n$ is called S-bounded if the Assumption (ii) is satisfied.

Theorem 3.1 [12]. *If the Assumptions* (i)–(iii) *are satisfied, then the system of mildly nonlinear equations $Ax = F(x)$ has a unique solution x^* in R^n.*

Let $B \succ 0$, we now use B-norm, i.e., $\|A\|_B = \|B^{\frac{1}{2}} A B^{-\frac{1}{2}}\|_2$. If $B = M - N$ is a symmetric P-regular splitting, we have $\|M^{-1}N\|_B < 1$. $\|\cdot\|_B$ is a compatible norm and has the following properties (see [1,9]).

Lemma 3.2. $\rho(A) \leqslant \|A\|_B$ *and* $\|AC\|_B \leqslant \|A\|_B \|C\|_B$.

Lemma 3.3. *Assume that $A = M - N$ is a symmetric P-regular splitting. If the Assumptions* (i)–(iii) *are satisfied, then* $\|M^{-1}(N + S(x, y))\|_M \leqslant \|M^{-1}(N + S)\|_M < 1$, $\forall x, y \in R^n$.

Proof. From the definition of B-norm, for $\forall x, y \in R^n$, we have

$$\|M^{-1}(N + S(x, y))\|_M = \|M^{-\frac{1}{2}}(N + S(x, y))M^{-\frac{1}{2}}\|_2.$$

From Assumption (i) we have $-S \preceq S(x, y) \preceq S$. Therefore, it holds that

$$M^{-\frac{1}{2}}(N - S)M^{-\frac{1}{2}} \preceq M^{-\frac{1}{2}}(N + S(x, y))M^{-\frac{1}{2}} \preceq M^{-\frac{1}{2}}(N + S)M^{-\frac{1}{2}},$$

which straightforwardly implies that

$$\|M^{-\frac{1}{2}}(N + S(x, y))M^{-\frac{1}{2}}\|_2 \leqslant \|M^{-\frac{1}{2}}(N + S)M^{-\frac{1}{2}}\|_2. \tag{3.1}$$

From the Assumptions (i) and (iii), $M - (N + S)$ is a symmetric P-regular splitting. Hence, we have

$$M^{-\frac{1}{2}}(N + S)M^{-\frac{1}{2}} \prec I. \tag{3.2}$$

From (3.1) and (3.2), the inequality $\|M^{-1}(N + S(x, y))\|_M \leqslant \|M^{-1}(N + S)\|_M < 1$ holds. We obtain this lemma. □

Theorem 3.4. *Assume that $A = M_i - N_i$, $i = 1, 2, \ldots, \alpha$ are symmetric P-regular splittings. Let $\sum_{i=1}^{\alpha} \|E_i\| = \eta$. If the Assumptions* (i)–(iii) *are satisfied, then there exists a positive integer m such that the iterative sequence $\{x^{(k)}\}$ generated by Method* 2.1 *converges to the solution x^* of* (2.1) *when $s(i, k) \geqslant m$, $i = 1, 2, \ldots, \alpha$.*

Proof. Let x^* be the solution of the mildly nonlinear system of equations (2.1). Then

$$M_i x^* = N_i x^* + F(x^*), \quad i = 1, 2, \ldots, \alpha. \tag{3.3}$$

Let $\varepsilon^{(i,k)} = x_i^{(k)} - x^*$, $i = 1, 2, \ldots, \alpha$. Then $\varepsilon^{(k)} = \sum\limits_{i=1}^{\alpha} E_i \varepsilon^{(i,k)}$. From (2.2) and (3.3) we have

$$M_i(y_i^{(j+1)} - x^*) = N_i(y_i^{(j)} - x^*) + F(y_i^{(j)}) - F(x^*), \quad i = 1, 2, \dots, \alpha. \tag{3.4}$$

By making use of (2.4) we can obtain

$$M_i(y_i^{(j+1)} - x^*) = (N_i + S(y_i^{(j)}, x^*))(y_i^{(j)} - x^*), \quad i = 1, 2, \dots, \alpha. \tag{3.5}$$

Thus, according to Method 2.1 we have

$$\varepsilon^{(i,k+1)} = H(i,k)\varepsilon^{(i,k)}, \quad i = 1, 2, \dots, \alpha$$

where $H(i,k) = \prod_{j=0}^{s(i,k)-1}(M_i^{-1}(N_i + S(y_i^{(j)}, x^*)))$, and

$$\varepsilon^{(k+1)} = H(k)\varepsilon^{(k)}, \tag{3.6}$$

where $H(k) = \sum_{i=1}^{\alpha} E_i H(i,k)$.

Since $\forall x \in R^n$, $\|(M_i^{-1}(N_i + S(x, x^*))\|_{M_i} \leqslant \|M_i^{-1}(N_i + S)\|_{M_i} < 1, \quad i = 1, 2, \dots, \alpha$, we obtain for $\forall x \in R^n$

$$\begin{aligned}\|H(i,k)\|_{M_i} &= \left\| \prod_{j=0}^{s(i,k)-1} (M_i^{-1}(N_i + S(y^{(j)}, x^*))) \right\|_{M_i} \\ &\leqslant \prod_{j=0}^{s(i,k)-1} \|M_i^{-1}(N_i + S(y^{(j)}, x^*))\|_{M_i} \\ &\leqslant \|M_i^{-1}(N_i + S)\|_{M_i}^{s(i,k)}, \quad i = 1, 2, \dots, \alpha.\end{aligned}$$

Thus, for $\forall x \in R^n$, it immediately follows that

$$\lim_{s(i,k)\to\infty} \|H(i,k)\|_{M_i} = 0, \quad i = 1, 2, \dots, \alpha.$$

From the norm equivalence theorem, for any compatible norm $\|\cdot\|$ we have

$$\lim_{s(i,k)\to\infty} \|H(i,k)\| = 0, \quad i = 1, 2, \dots, \alpha.$$

Therefore, for a real number θ satisfying $0 < \theta < \frac{1}{\eta}$, there exist positive intergers m_i, $i = 1, 2, \dots, \alpha$, independent of x and k, when $s(i,k) \geq m_i$, $i = 1, 2, \dots, \alpha$, we have

$$\|H(i,k)\| \leqslant \theta < \frac{1}{\eta}, \quad i = 1, 2, \dots, \alpha$$

Let $m = \max_{i=1}^{\alpha}\{m_i\}$, then when $s(i,k) \geq m$, $i = 1, 2, \dots, \alpha$, in the view of (3.6) we have

$$\|H(k)\| \leqslant \sum_{i=1}^{\alpha} \|E_i\|\|H(i,k)\| \leqslant \theta\eta < 1, \quad k = 1, 2, \dots,$$

and after that

$$\|\varepsilon^{(k+1)}\| = \|H(k)\varepsilon^{(k)}\| \leqslant \|H(k)\|\|\varepsilon^{(k)}\| \leqslant (\theta\eta)^k \|\varepsilon^{(1)}\|.$$

Thus, we obtain $\lim_{k\to\infty} \varepsilon^{(k+1)} = 0$. □

For the general compatible norm we obtain the convergence Theorem 3.4 for Method 2.1, but m is theoretical. In order to obtain the lower bound of m, we use B-norm.

Lemma 3.5. *Let $A \succ 0$. Assume that $A = M - N$ is a symmetric P-regular splitting. If there exist two symmetric matrices S_1 and S_2 such that $0 \preceq S_1 \preceq S_2$ and $A - S_1 \succ 0$, $A - S_2 \succ 0$, then $\|M^{-1}(N + S_1)\|_{A-S_1} \leq \|M^{-1}(N + S_2)\|_{A-S_2} < 1$.*

Proof. From the definition of B-norm, we have

$$\begin{aligned}
\|M^{-1}(N + S_1)\|_{A-S_1} &= \|(A - S_1)^{\frac{1}{2}} M^{-1}(N + S_1)(A - S_1)^{-\frac{1}{2}}\|_2 \\
&= \|I - (A - S_1)^{\frac{1}{2}} M^{-1}(A - S_1)^{\frac{1}{2}}\|_2 \\
&= \lambda_{\max}(I - (A - S_1)^{\frac{1}{2}} M^{-1}(A - S_1)^{\frac{1}{2}}) \\
&= \lambda_{\max}(I - M^{-1}(A - S_1)) \\
&= \lambda_{\max}(M^{-1}(N + S_1)) \\
&= \lambda_{\max}(M^{-\frac{1}{2}}(N + S_1)M^{-\frac{1}{2}})
\end{aligned}$$

because $0 \preceq S_1 \preceq S_2$ implies $0 \preceq M^{-\frac{1}{2}}(N + S_1)M^{-\frac{1}{2}} \preceq M^{-\frac{1}{2}}(N + S_2)M^{-\frac{1}{2}}$, we have

$$\lambda_{\max}(M^{-1}(N + S_1)) \leqslant \lambda_{\max}(M^{-1}(N + S_2)).$$

Therefore, the inequality $\|M^{-1}(N + S_1)\|_{A-S_1} \leqslant \|M^{-1}(N + S_2)\|_{A-S_2}$ holds. On the other hand, because $A - S_2 = M - (N + S_2)$ is a symmetric P-regular splitting, $\lambda_{\max}(M^{-1}(N + S_2)) < 1$ holds. Therefore, we obtain

$$\|M^{-1}(N + S_1)\|_{A-S_1} \leq \|M^{-1}(N + S_2)\|_{A-S_2} < 1.$$

This lemma is obtained. □

Theorem 3.6. *Assume that $A = M_i - N_i$, $i = 1, 2, \ldots, \alpha$ are symmetric P-regular splittings. Let $A = M - N$ be such that $M \succeq M_i$, $i = 1, 2, \ldots, \alpha$ holds. Let $\sum_{i=1}^{\alpha} \|E_i\|_{A-S} = \eta$ and $\rho(M^{-1}(N + S)) = r$. If the Assumptions* (i)–(iii) *are satisfied, then the iterative sequence $\{x^{(k)}\}$ generated by Method* 2.1 *converges to the solution x^* of* (2.1) *when $s(i, k) \geqslant [-\log_r \eta] + 2$.*

Proof. From Theorem 3.4, the iterative sequence $\{x^{(k)}\}$ generated by the Method 2.1 converges to the solution x^* of (2.1) if there exists a real number $0 < \theta < \frac{1}{\eta}$ such that

$$\|H(i,k)\| \leqslant \theta\eta, \quad i = 1, 2, \ldots, \alpha$$

holds. From Lemma 3.5 and $N \succeq 0$, for $i = 1, 2, \ldots, \alpha$, we have

$$\begin{aligned}\|M_i^{-1}(N_i + S(x, x^*))\|_{A-S(x,x^*)} &= \lambda_{\max}(M^{-1}(N + S(x, x^*))) \\ &\leqslant \lambda_{\max}(M^{-1}(N + \langle S(x, x^*)\rangle)) \\ &= \|M_i^{-1}(N_i + \langle S(x, x^*)\rangle)\|_{A-\langle S(x,x^*)\rangle}.\end{aligned}$$

From Assumptions (ii) and (iii), for $i = 1, 2, \ldots, \alpha$, we have

$$\begin{aligned}\|M_i^{-1}(N_i + \langle S(x, x^*)\rangle)\|_{A-\langle S(x,x^*)\rangle} &\leqslant \|M_i^{-1}(N_i + S)\|_{A-S} \\ &= \lambda_{\max}(I - M_i^{-1}(A - S)) \\ &= \lambda_{\max}(I - (A - S)^{\frac{1}{2}} M_i^{-1}(A - S)^{\frac{1}{2}}) \\ &\leqslant \lambda_{\max}(I - (A - S)^{\frac{1}{2}} M^{-1}(A - S)^{\frac{1}{2}}) \\ &= \lambda_{\max}(I - M^{-1}(A - S)) \\ &= \lambda_{\max}(M^{-1}(N + S)) = r < 1.\end{aligned}$$

Thus, for $i = 1, 2, \ldots, \alpha$, we obtain

$$\begin{aligned}\|H(i,k)\|_{A-S} &= \|\prod_{j=0}^{s(i,k)-1}(M_i^{-1}(N_i + S(y^{(j)}, x^*)))\|_{A-S} \\ &\leqslant \prod_{j=0}^{s(i,k)-1}\|M_i^{-1}(N_i + S(y^{(j)}, x^*))\|_{A-S} \leqslant r^{s(i,k)}.\end{aligned}$$

Therefore, $\|H(i,k)\|_{A-S} \leqslant \theta = \frac{r}{\eta}$, $i = 1, 2, \ldots, \alpha$ if

$$r^{s(i,k)} \leqslant \frac{r}{\eta}, \quad i = 1, 2, \ldots, \alpha.$$

Thus, we obtain $s(i,k) \geqslant -\log_r \eta + 1$, $i = 1, 2 \ldots, \alpha$. Therefore, when $s(i,k) \geqslant [-\log_r \eta] + 2$, $i = 1, 2, \ldots, \alpha$, the iterative sequence $\{x^{(k)}\}$ generated by Method 2.1 converges to the solution x^* of (2.1). $\square$

Remark 3.1. If $S = 0$, Theorem 3.6 is an improvement of Corollary 2.3 in [10] and Theorem 3.2 in [13].

Example 3.1. Let $Ax = bh^2 g(x)$, $x \in [0, 1]$ with

$$A = (a_{i,j})_{n\times n} = \begin{cases} -1, & j = i - 1, \\ 3, & j = i, \\ -1, & j = i + 1. \end{cases}$$

$$b = 0.1,$$

$$g(x) = (\sin(x_1 + x_2), \ldots, \sin(x_{i-1}+x_i+x_{i+1}), \ldots, \sin(x_{n-1}+x_n), \sin x_n)^{\mathrm{T}}.$$

Let $h = \frac{1}{n} = 0.02$, we can obtain from (2.4) and property of rank-one matrix the matrix $S = 0.185bI \succeq \langle S(x, y)\rangle$. Let $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3$ with

$$M_i = \begin{pmatrix} M_{i1} & 0 & 0 \\ 0 & M_{i2} & 0 \\ 0 & 0 & M_{i3} \end{pmatrix}, \quad i = 1, 2, 3.$$

$$M_{11} = \begin{pmatrix} 3.8 & -1.4 & & & \\ -1.4 & 3.8 & -1.4 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.4 & 3.8 & -1.4 \\ & & & -1.4 & 4.4 \end{pmatrix}_{15\times 15},$$

$$M_{12} = \begin{pmatrix} 4.4 & -1.4 & & & \\ -1.4 & 3.8 & -1.4 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.4 & 3.8 & -1.4 \\ & & & -1.4 & 4.4 \end{pmatrix}_{15\times 15},$$

$$M_{13} = \begin{pmatrix} 4.4 & -1.4 & & & \\ -1.4 & 3.8 & -1.4 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.4 & 3.8 & -1.4 \\ & & & -1.4 & 3.8 \end{pmatrix}_{20\times 20},$$

$$M_{21} = \begin{pmatrix} 3.4 & -1.2 & & & \\ -1.2 & 3.4 & -1.2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.2 & 3.4 & -1.2 \\ & & & -1.2 & 4.2 \end{pmatrix}_{15\times 15},$$

$$M_{22} = \begin{pmatrix} 4.2 & -1.2 & & & \\ -1.2 & 3.4 & -1.2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.2 & 3.4 & -1.2 \\ & & & -1.2 & 4.2 \end{pmatrix}_{15\times 15},$$

$$M_{23} = \begin{pmatrix} 4.2 & -1.2 & & & \\ -1.2 & 3.4 & -1.2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.2 & 3.4 & -1.2 \\ & & & -1.2 & 3.4 \end{pmatrix}_{20\times 20},$$

$$M_{31} = \begin{pmatrix} 3.2 & -1.1 & & & \\ -1.1 & 3.2 & -1.1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.1 & 3.2 & -1.1 \\ & & & -1.1 & 4.1 \end{pmatrix}_{15\times 15},$$

$$M_{32} = \begin{pmatrix} 4.1 & -1.1 & & & \\ -1.1 & 3.2 & -1.1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.1 & 3.2 & -1.1 \\ & & & -1.1 & 4.1 \end{pmatrix}_{15\times 15},$$

$$M_{33} = \begin{pmatrix} 4.1 & -1.1 & & & \\ -1.1 & 3.2 & -1.1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.1 & 3.2 & -1.1 \\ & & & -1.1 & 3.2 \end{pmatrix}_{20\times 20}.$$

Let

$$E_1 = \operatorname{diag}(\underbrace{0, \ldots, 0}_{15}, \underbrace{0.5, \ldots, 0.5}_{15}, \underbrace{0.4, \ldots, 0.4}_{20}),$$

$$E_2 = \operatorname{diag}(\underbrace{0.45, \ldots, 0.45}_{15}, \underbrace{0, \ldots, 0}_{15}, \underbrace{0.6, \ldots, 0.6}_{20}),$$

$$E_3 = \operatorname{diag}(\underbrace{0.55, \ldots, 0.55}_{15}, \underbrace{0.5, \ldots, 0.5}_{15}, \underbrace{0, \ldots, 0}_{20}).$$

Let $M = M_1$, we have $M \succeq M_i$, $i = 1, 2, 3$, then $\rho(M^{-1}(N + S)) < 0.4$ and $\sum_{i=1}^{3} \|E_i\|_{A-S} < 8.25$. We obtain from Theorem 3.6 that $s(i, k) \geqslant [-\log_{0.4} 8.25] + 2 = 4$ holds.

Remark 3.2. It is obvious that m mainly depend on r, if r approaches to one, m will be large. Hence, the splitting $A - S = M - (N + S)$ is very important. Thus, the splitting $A = M - N$ is the key factor to m.

Acknowledgements

The authors are very much indebted to professor M. Neumann and the referee for their comments and suggestions which greatly improve the original manuscript of this paper.

References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.
[2] J.M. Ortega, W.C. Rheinboldt, Iterative Solutions of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.
[3] Z.Z. Bai, A class of two-stage iterative methods for systems of weakly nonlinear equations, Numer. Algorith. 14 (1997) 295–319.
[4] Z.Z. Bai, Parallel multisplitting two-stage iterative methods for large sparse systems of weakly nonlinear equations, Numer. Algorith. 15 (1997) 347–372.
[5] Z.Z. Bai, V. Migallón, J. Penadés, D.B. Szyld, Block and asynchronous two-stage methods for mildly nonlinear systems, Numer. Math. 82 (1999) 1–20.
[6] D.P. O'Leary, R.E. White, Multi-splittings of matrices and parallel solution of linear systems, SIAM J. Alg. Dis. Mec. 6 (1985) 630–640.
[7] Z.-H. Cao, Nonstationary two-stage multisplitting methods with overlapping blocks, Linear Algebra Appl. 285 (1998) 153–163.
[8] Z.-H. Cao, Z.-Y. Liu, Symmetric multisplitting of a symmetric positive definite matrix, Linear Algebra Appl. 285 (1998) 309–319.
[9] M.J. Castel, V. Migallón, J. Penadés, Convergence of non-stationary multisplitting methods for Hermitian positive definite matrices, Math. Comput. 67 (1998) 209–220.
[10] V. Migallón, J. Penadés, D.B. Szyld, Nonstationary multisplittings with general weighting matrices, SIAM J. Matrix Anal. Appl. 22 (2001) 1089–1094.
[11] C.L. Wang, Z.Z. Bai, Sufficient conditions for the convergent splittings of non-Hermitian positive definite matrices, Linear Algebra Appl. 330 (2001) 215–218.
[12] Z.Z. Bai, C.L. Wang, Convergence theorems for parallel multisplitting two-stage iterative methods for mildly nonlinear systems, Linear Algebra Appl. 362 (2003) 237–250.
[13] C.L. Wang, Nonstationary multisplitting with general weighting matrices for non-Hermitian positive definite systems, Appl. Math. Lett. 16 (2003) 919–924.
[14] J.M. Ortega, Introduction to Parallel and Vector Solution of Linear Systems, Plenum Press, New York, 1988.
[15] R.E. White, Multisplitting of a symmetric positive definite matrix, SIAM J. Matrix Anal. Appl. 11 (1990) 69–82.